2023 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 1–2 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

1. A frog is hopping along a line of 2000 stones, numbered in order. The frog begins on stone #17 and hops forward the same number of stones with each hop. Suppose that the frog lands on stone #353 and stone #1081 during its trip. Compute the greatest possible value for the number of stones the frog moves forward on each hop.

2. Consider the increasing sequence $2, 3, 20, 22, 23, 30, \ldots$ consisting of all positive integers that can be formed from the digits 0, 2, and 3. Compute the number of digits in the 2023^{rd} entry in the sequence (2 is the first entry, 3 is the second entry, and so on).

ANSWER TO PROBLEM 1

ANSWER TO PROBLEM 2

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Problems 3–4 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

3. Observe that a square whose sides have length 4 has the same area and perimeter. Compute the length of the side of a regular octagon whose area and perimeter are equal.

4. Compute the number of sequences that are arrangements of the numbers 1, 2, 3, 4, 5, 6, 7 and satisfy the requirement that no term in the sequence is larger than the square of the term that follows it.

ANSWER TO PROBLEM 3

ANSWER TO PROBLEM 4

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Problems 5–6 **Name** ______________________________

Time Limit: 10 minutes **School** ______________________________

5. Let $ABCD$ be a square and let P and Q be the midpoints of sides $\overline{AB}$ and $\overline{BC}$, respectively. Compute $\sin(\mathrm{m}\angle PDQ)$. Express your answer as a fraction in simplest form.

6. Compute the number of distinct ordered pairs of real numbers (x, y) that satisfy at least two of the equations

$$\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \\ x + y &= 9 \\ 2x + 2y &= 10. \end{aligned}$$

ANSWER TO PROBLEM 5

ANSWER TO PROBLEM 6

2023 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 7–8 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

7. A country has 2023 cities, and is trying to build roads connecting some cities to other cities. However, building roads is expensive, so the country's president wants to minimize the number of roads that need to be built. Furthermore, the president wants to ensure that if any two roads are closed, citizens are still able to travel from any city to any other city. Compute the minimum number of roads needed to satisfy these conditions.

8. In $\triangle ABC$, $BC = 7$, $AC = 5$, and $AB = 6$. Let X and Y be points on line $\overleftrightarrow{BC}$ such that line $\overleftrightarrow{AX}$ is the angle bisector of angle $\angle BAC$ and line $\overleftrightarrow{AY}$ is perpendicular to line $\overleftrightarrow{AX}$. Compute the area of $\triangle ACY$.

ANSWER TO PROBLEM 7

ANSWER TO PROBLEM 8

2023 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 9–10 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

9. A contestant is playing a game on a game show. She gets to spin a spinner five times, which could land on each of the numbers 0, 1, 3, or 6 with equal probability. She will win if the total of her spins is 20 or more. The spins are independent. Compute the probability that she wins the prize.

10. Compute the least positive real number k such that the graphs of $y = kx$ and $y = x \cdot \lfloor x \rfloor$ have an intersection of length greater than 10. (Note: The notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

ANSWER TO PROBLEM 9

ANSWER TO PROBLEM 10

2023 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 11–12 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

11. For $n \geq 1$, let $S(n) = i + i^2 + \cdots + i^n$, where $i = \sqrt{-1}$. Compute $\displaystyle\sum_{n=1}^{2023} S(n)$.

12. Four identical orange candies and five identical blueberry candies are to be distributed to three (distinct) children so that each child gets at least one candy (of either type). Compute the number of ways in which this can be done.

ANSWER TO PROBLEM 11

ANSWER TO PROBLEM 12

2023 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 13–14 **Name** ______________________________

Time Limit: 10 minutes **School** ______________________________

13. Let $q(x) = x^3 - 3x - 1$. Let a, b, and c be distinct real numbers such that ab, bc, and ca are roots of $q(x)$. Let $p(x)$ be the cubic polynomial with leading coefficient 1 whose roots are a, b, and c. Compute all possible values of $p(2)$.

14. Compute the number of integer bases, n, less than 20, such that $n!_n$ (that is, $n!$ expressed in base n) ends in at least two zeroes.

ANSWER TO PROBLEM 13

ANSWER TO PROBLEM 14

Problems 15–16 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

15. Olivia has two \$20 bills and three \$10 bills. She flips a fair coin for each bill. If the coin comes up heads she places the bill into pile A, while if it comes up tails she places the bill into pile B. Compute the expected number of bills in the pile of greater value.

16. Forgetful Farhad forgot many of the facts about the quadrilateral $ARML$ that he had been studying in class. For instance, he forgot all the angles, though he did remember noticing that the average of angles $\angle A$ and $\angle M$ was the same as the average of the angles $\angle R$ and $\angle L$. He did remember that the sides were 3, 4, 8, and 9, but he forgot which side had which length. He also forgot the area of $ARML$. Compute it for him.

ANSWER TO PROBLEM 15

ANSWER TO PROBLEM 16

2023 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 17–18 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

17. The fraction $\frac{1}{6^2}$ has a repeating base-seven expansion. Compute the least number of digits that repeat.

18. Complete the cross-number puzzle below, where each Across answer is a four-digit number and each Down answer is a three-digit number. No answer begins with the digit 0.
Your answer must be written in the space at the bottom of this page, not the grid to the right!

Across
1. Shares a factor greater than 1 with 2023
5. Digits form an increasing arithmetic progression from left to right
6. A base-four integer with distinct digits

Down
1. A perfect square
2. The least integer $N > 500$ such that $20^{23} \cdot N$ is a perfect cube
3. Twice a perfect square
4. A palindrome

1	2	3	4
5			
6			

ANSWER TO PROBLEM 17

ANSWER TO PROBLEM 18

1	2	3	4
5			
6			

Problems 19–20 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

19. A fair coin is flipped until two tails in a row appear. Compute the probability that two heads in a row were flipped at some point.

20. Suppose that θ is a real number such that

$$\sqrt[4]{1+\sin(\theta)}+\sqrt[4]{1-\sin(\theta)}=\sqrt{2}.$$

Compute $|\cos(\theta)+\sec(\theta)|$.

ANSWER TO PROBLEM 19

ANSWER TO PROBLEM 20

2023 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 21–22 **Name** ______________________________

Time Limit: 10 minutes **School** ______________________________

21. Compute the number of integers x that satisfy $\lfloor (x+0.001)^2 - x^2 \rfloor = 2023$.

22. Given that $2^{8^x} = 8^{2^x}$, there is a unique solution for x in the positive real numbers. This solution can be written as $x = \log_a(b)$ where a and b are integers and $a < 15$. Compute the ordered pair (a, b).

ANSWER TO PROBLEM 21

ANSWER TO PROBLEM 22

2023 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 23–24 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

23. All five-digit numbers that contain no digit of 9 and no digit of 0 are written in increasing order. Compute the 11,923$^{\text{rd}}$ number on this list.

24. A cube has three of its vertices at the points $(2, 5, 1)$, $(10, 10, 4)$, and $(11, 1, 0)$. Compute the surface area of the cube.

ANSWER TO PROBLEM 23

ANSWER TO PROBLEM 24

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Part I Answers

1.	56	2.	7
3.	$4\sqrt{2}-4$	4.	360
5.	$\frac{3}{5}$ (must be this form)	6.	3
7.	3035	8.	$30\sqrt{6}$
9.	$43/512$	10.	10
11.	$-1012+1012i$	12.	228
13.	-3 and 19 (either order)	14.	9

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Part II Answers

15. $27/8$ (or equivalent)

16. $12\sqrt{6}$

17. 6

18.

1	5	1	3
2	4	6	8
1	0	2	3

19. 0.5 (or equivalent)

20. 14

21. 500

22. $(4, 3)$ (must be this ordered pair)

23. 38,333

24. 294

Tryout Results 2023

Item Analysis

($n = xxx$ for Q1–Q24)

Q	**# right**	**%**
1		%
2		%
3		%
4		%
5		%
6		%
7		%
8		%
9		%
10		%
11		%
12		%

Q	**# right**	**%**
13		%
14		%
15		%
16		%
17		%
18		%
19		%
20		%
21		%
22		%
23		%
24		%

Distribution of Top Scores

Score	$\boldsymbol{n}$
24	
23	
22	
21	
20	
19	
18	
17	
16	
15	
14	