Chicago Area All-Star Math Team

2023 Tryout Solutions

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1. From stone #17 to stone #353 is $353 - 17 = 336$ stones. From #353 to #1081 is $1081 - 353 = 728$ stones. Since the frog always jumps the same number of stones, that number must divide evenly into both 336 and 728. The largest number that does so is their greatest common divisor. Quickly factoring, $336 = 2^4 \cdot 3 \cdot 7$ and $728 = 2^3 \cdot 7 \cdot 13$ so their greatest common divisor is $2^3 \cdot 7 = \boxed{56}$. (SG)

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2. The leading digit of a number in the sequence can only be 2 or 3, but any other digit could be 0, 2, or 3. Thus there are 2 1-digit numbers in the sequence, $2 \cdot 3 = 6$ 2-digit numbers, $2 \cdot 3^2 = 18$ 3-digit numbers, and $2 \cdot 3^{k-1}$ k-digit numbers in the sequence for general k. Counting all the numbers in the sequence with up to n digits, the result is $2 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n-1}$. This is a geometric series whose sum is $2\frac{1-3^n}{1-3} = 3^n - 1$. Thus, we need to find n so that $3^{n-1} - 1 < 2023 \le 3^n - 1$. A little work shows that $3^6 = 729$ and $3^7 = 2187$, so the 2023^{rd} entry in the sequence has $\boxed{7}$ digits. (CJ)

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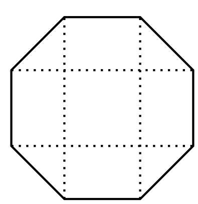
3. An octagon may be subdivided into smaller shapes as shown below:



If the side-length of the octagon is s, then the perimeter is $8s$. The area can be found by adding the pieces shown above. The central square has area s^2. The four triangles are isosceles right triangles with hypotenuse of length s, so the sides are $s/\sqrt{2}$. These four triangles combine together to form a square of side-length s and area s^2. The four rectangles have sides s and $s/\sqrt{2}$, so their total area is $4 \cdot s \cdot s/\sqrt{2} = s^2 \cdot 2\sqrt{2}$. Thus, the area of the octagon is $s^2(2 + \sqrt{2})$. Setting this equal to $8s$ and solving for s gives $s = \frac{8}{2+\sqrt{2}}$ which rationalizes to $\boxed{4\sqrt{2} - 4}$. (MF)

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4. Since $1^2 = 1$ is smaller than any of the other numbers, 1 must come first in the sequence. The 2 can only follow the 1, the 3, or the 4. Those are the only two restrictions, as $3^2 = 9$ and certainly the squares of the larger numbers are larger than any of the numbers 1–7. So arrange the numbers 3, 4, 5, 6, 7 ($5! = 120$ ways), prepend 1 to the sequence, and then insert the 2 after either the 1, the 3, or the 4 (3 choices). This brings the total number of possible arrangements to $3 \cdot 120 = \boxed{360}$. (JR, modified by MF)

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5. One approach is note that $\triangle QDC$ is a 1–2–$\sqrt{5}$ right triangle, so $\sin(\text{m}\angle QDC) = \frac{1}{\sqrt{5}}$ and $\cos(\text{m}\angle QDC) = \frac{2}{\sqrt{5}}$. Then $\text{m}\angle BDQ = 45^\circ - \text{m}\angle QDC$, so the angle addition formulas may be used to find the sine and cosine of $\text{m}\angle BDQ$. Then the double-angle formula completes the problem.

A closely related approach would be to note that $\text{m}\angle PDQ = 90^\circ - 2\text{m}\angle QDC$ so $\sin(\text{m}\angle PDQ) = \sin(90^\circ - 2\text{m}\angle QDC) = \cos(2\text{m}\angle QDC)$ and use the double-angle formula for cosine and the values found above for the sine and cosine of $\text{m}\angle QDC$.

A nifty trick is as follows. Let the sides of square $ABCD$ have length 2. Then the lengths $PD = QD = \sqrt{5}$ from the Pythagorean Theorem. The area of the square is 4, the triangles $\triangle APD$ and $\triangle CQD$ have area 1 and $\triangle PBQ$ has area $\frac{1}{2}$. Subtracting, the area of $\triangle PDQ$ is found to be $\frac{3}{2}$. For any triangle, its area is $\frac{1}{2}xy\sin(Z)$ where Z is the angle between the sides with length x and y. In this case, $\frac{3}{2} = \frac{1}{2}\sqrt{5}^2\sin(\text{m}\angle PDQ)$ revealing that $\sin(\text{m}\angle PDQ) = \boxed{\frac{3}{5}}$. (MC2)

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6. There are $\binom{4}{2} = 6$ pairs of equations, but the final two are equations of parallel lines so they cannot have a common solution. Aside from these two, the slopes of the lines are all different, so each other pair leads to a solution. However, the first, second, and fourth equations have $(3, 2)$ as a common solution, so altogether there are only $\boxed{3}$ distinct ordered pairs that are solutions to at least two of the equations. (HM)

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7. There can be no city with only two roads connecting it to other cities, because if those two roads were closed the citizens could not get to that city from anywhere else. So every city must have at least three roads that have one end there. That means there are at least 6069 ends of the roads. But each road has two ends, so the total number of road ends must be even. So there are actually at least 6070 ends, meaning that there are at least $\boxed{3035}$ roads.

There are many arrangements with exactly 3035 roads that satisfy the requirements. One such is to let the cities be numbered $C_1, C_2, \ldots, C_{2022}$ and C representing the capital city.

Build 1011 roads from C_1 to C_3 to C_5 to ... to C_{2021} back to C_1 (basically, a circular road connecting all the odd cities), and another 1011 roads connecting all the even cities in a loop. Build another 1009 roads connecting C_1 to C_2, C_3 to C_4, and so on, connecting each odd city the next higher even number, until connecting C_{2017} to C_{2018}. Finally, build four more roads connecting C to each of last four other cities, C_{2019} through C_{2022}. Altogether this is $1011 + 1011 + 1009 + 4 = 3035$ roads, and closing any two roads still leaves it possible to get from any city to any other. (BB)

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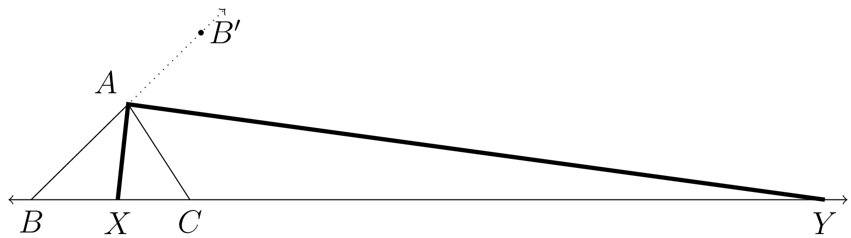
8. The figure is sketched below:



Point B' lies on the continuation of $\overrightarrow{BA}$ past point A. Line $\overleftrightarrow{AY}$ is the bisector of exterior angle $\angle B'AC$ to $\triangle ABC$, so by the exterior bisector theorem $YC/YB = AC/AB = 5/6$. (The exterior bisector theorem essentially says the same thing as the angle bisector theorem for triangles, only it applies to the exterior angles of the triangle and the lengths of the extensions of the opposite side to where it meets this bisector.) Since $YB = YC + 7$ the length $YC = 35$. Note that $\triangle ACY$ and $\triangle ABC$ have the same height while the base of the former is five times the base of the latter. So the area of $\triangle ACY$ is five times that of $\triangle ABC$. The area of $\triangle ABC$ can be computed by Heron's formula: $\sqrt{9(9-7)(9-6)(9-5)} = 6\sqrt{6}$. So the area of $\triangle ACY$ is $\boxed{30\sqrt{6}}$. (MC1)

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9. The winning spin combinations are 66666 (which can only be achieved one way), 66663 (since the 3 could be in any of five places, this combination can be achieved 5 ways), 66661 (5 ways), 66660 (5 ways), 66633 (10 ways), 66631 (20 ways), 66630 (20 ways), 66611 (10 ways), and 66333 (10 ways). So altogether she has 86 winning combinations. There are $4^5 = 1024$ possible ways for the spins to come out, so her probability of winning is $86/1024 = \boxed{43/512}$. (MF)

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10. If k is not an integer, then since $\lfloor x \rfloor$ is an integer the graphs will only intersect at $x = 0$ and have no length in common. On the other hand, if k is an integer, then the two graphs are the same as long as $\lfloor x \rfloor = k$, or $k \leq x < k+1$. Thus, the graphs intersect in the line segment between (k, k^2) and $(k+1, (k+1)k)$ which has a length of $\sqrt{1+k^2}$ by using the distance formula. For positive k, this is at least 10 when $k \geq \sqrt{99}$. Since k must also be an integer, the value needed here is $\boxed{10}$. (MC1)

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11. Probably the easiest approach it to start finding the values of $S(n)$: $S(1) = i$, $S(2) = -1 + i$, $S(3) = -1$, $S(4) = 0$, and these four values then repeat forever. So each four terms of the sum add to $-2 + 2i$. There are 505 groups of the four terms, plus $S(2021)$, $S(2022)$, and $S(2023)$, giving a total of $505(-2+2i)+i+(-1+i)+(-1) = \boxed{-2012 + 2012i}$. (MC1)

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12. Using a sticks-and-stones argument, there are $\binom{6}{2} = 15$ ways to split the orange candies and $\binom{7}{2} = 21$ ways to split the blueberry candies. So altogether there are $15 \cdot 21 = 315$ ways to split the candies. But this counts divisions where one or even two of the children don't get any candy!

How many ways are there that the first child gets no candy? Split the remaining candy between the other two children: $\binom{5}{1} = 5$ ways to split the orange and 6 ways to split the blueberry, so 30 possibilities. Since any of the three children might have been the one to receive no candy, that makes for $3 \times 30 = 90$ divisions. But this double-counts cases where *two* children get no candy and one gets all of it! So by the principle of inclusion-exclusion, the 3 possibilities where one child gets all the candy need to be discounted. So there are really $90 - 3 = 87$ divisions in which at least one child gets no candy.

Sin each child must receive at least one candy, take the number of possible distributions and subtract those where at least one child gets no candy: $315 - 87 = \boxed{228}$. (MC1)

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13. Notice that the product $(ab)(bc)(ca) = 1$ because it is the product of the roots of $q(x)$, which is given by the opposite of its constant term. Thus $(abc)^2 = 1$ so $abc = \pm 1$. Then $a = \frac{abc}{bc}$, $b = \frac{abc}{ac}$ and $c = \frac{abc}{ab}$ are either the reciprocals of the roots of $q(x)$ or their opposites.

Now given any polynomial, to produce a polynomial whose roots are the reciprocals of the original polynomial one simply reverses the order of the coefficients. For example, a polynomial whose roots are the reciprocals of the roots of $3x^4 - 2x^3 + 9x^2 - 5$ is $-5x^4 + 9x^2 - 2x + 3$. You should convince yourself of this fact by considering the complete factorization of the polynomial.

So a polynomial whose roots are a, b, and c, the reciprocals of the roots of $q(x)$, is $-x^3 - 3x^2 + 1$. The problem requires the leading coefficient to be 1, so one possibility for $p(x)$ is $p(x) = x^3 + 3x^2 - 1$, and $p(2) = 19$ in this case. To find a polynomial whose roots are the opposites of a, b, and c (that is, whose roots are the opposites of the reciprocals of those of $q(x)$), change the signs of the $n-1$, $n-3$, and so on terms: $p(x) = x^3 - 3x^2 + 1$, making $p(2) = -3$. So the possible values for $p(2)$ are $\boxed{-3 \text{ and } 19}$. (BB, as modified by MF)

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14. Expressed in base n, a number ends in at least two zeroes if and only if it is divisible by n^2. So the problem is asking for the number of integers n in the range $2 \leq n < 20$ for which $n!$ is divisible by n^2.

This never happens for primes, so 2, 3, 5, 7, 11, 13, 17, and 19 are eliminated. $4! = 24 = 120_4$ so it is also eliminated. It is routine, if tedious, to check that all the other numbers do satisfy the condition: the list 6, 8, 9, 10, 12, 14, 15, 16, and 18 comprises $\boxed{9}$ numbers. (MC1, modified by MF)

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15. Since the total amount of money Olivia is holding is \$70 but all the individual bills' amounts are divisible by \$10, there is no way in which she can distribute the money into two piles that have the same amount of money in each, so ties need not be considered in this problem. The table below lists the possible bills that could be in the larger-valued pile (T represents a \$20 bill, and X a \$10 bill), the number of outcomes of the coin flips that could create that pile, the number of bills in that pile, and the total contribution to the expected value (which is the number of bills times the number of ways to get that many bills for this combination):

Bills in pile	Number of outcomes	Number of bills	Contribution
TT	2	2	4
TTX	6	3	18
TTXX	6	4	24
TTXXX	2	5	10
TXX	12	3	36
TXXX	4	4	16
Totals:	32		108

For instance, the combination TTX can occur in 6 ways, because both of the \$20 coin flips must land the same way (two choices: heads or tails) and one of the three \$10 coin flips must land that way, too, so there are $2 \times 3 = 6$ coin flip combinations that result in this pile being the larger-valued pile.

Expected value is the sum of (values times number of combinations to get that value) divided by the total number of combinations. In this case, the expected value is $108/32 = \boxed{27/8}$. (JR, modified by MF)

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16. Notice that since the pairs of opposite angles have the same average, they also have the same sum. The only way that can happen is if both pairs add to 180°, making $ARML$ a cyclic quadrilateral. Brahmagupta's formula can then be applied to compute the area. The semiperimeter is $\frac{3+4+8+9}{2} = 12$, so the area is $\sqrt{(12-3)(12-4)(12-8)(12-9)} = \sqrt{9 \cdot 8 \cdot 4 \cdot 3} = \boxed{12\sqrt{6}}$. (MF)

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17. Compute the base-seven representation of $1/6^2 = 1/36$ by repeatedly multiplying by 7, taking out the whole part and keeping the fraction, until the numbers start to repeat:

$$\begin{aligned}
7 \times 1/36 &= 7/36 \\
7 \times 7/36 &= 49/36 = 1\ 13/36 \\
7 \times 13/36 &= 91/36 = 2\ 19/36 \\
7 \times 19/36 &= 133/36 = 3\ 25/36 \\
7 \times 25/36 &= 175/36 = 4\ 31/36 \\
7 \times 31/36 &= 217/36 = 6\ 1/36
\end{aligned}$$

So in base-seven, $1/36 = 0.\overline{012346}$ and the length of the repeating block is $\boxed{6}$. (HM)

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18. 2-Down seems to be a good place to start. $20^{23} = 2^{46}5^{23}$ so N will need to supply two factors of 2 and one factor of 5 to arrive at a perfect cube. That is, N is $20c$ where c is a cube. The smallest choice for c so that $N > 500$ is $c = 27$ so $N = 540$.

Now since the second digit of 5-Across is 4, and the digits must be an increasing arithmetic progression, 5-Across is either 3456 or 2468. 3456 is ruled out by 1-Down, because no 3-digit squares have a 3 in the tens-digit. So 5-Across is 2468. 1-Down is consequently 121, 225, 324, 529, 625, or 729.

But the final digit in 1-Down must be a legitimate base-four digit, so 1-Down can only be 121.

Now the last digit of 3-Down is even, and the digits 1 and 0 have already been used in 6-Across, so the only remaining base-four even digit is 2. Thus 6-Across is 1023. 4-Down is now known to be 383.

1-Across is 15x3 for some digit x. Factor 2023: $2023 = 7 \cdot 17^2$. Checking the numbers 1503, 1513, …, 1593, only 1533 is divisible by 7, while 1513 is divisible by 17. Of these possibilities, checking with 3-Down, 162 is twice a square but 362 is not.

With these numbers in place, the completed puzzle is

1	5	1	3
2	4	6	8
1	0	2	3

. (CJ)

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19. A careful case analysis could be performed based on whether the previous flip was a heads or a tails, but this problem is much simpler than that. All that it is asking is whether two heads in a row are flipped before two tails in a row, or vice versa. Since these two events are equally likely, each is equally likely to occur before the other. Thus, the probability is $\boxed{0.5}$. (HM)

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20. Start by squaring the given expression: $\sqrt{1-\sin(\theta)} + 2\sqrt[4]{(1-\sin(\theta)(1+\sin(\theta))} + \sqrt{1+\sin(\theta)} = 2$.

Now $(1-\sin(\theta)(1+\sin(\theta)) = 1-\sin^2(\theta) = \cos^2(\theta)$. So the middle term on the left-hand side of the equation above simplifies to $2\sqrt{|\cos(\theta)|}$. Subtract this from both sides of the equation and square:

$$(1-\sin(\theta)) + 2\sqrt{(1-\sin(\theta)(1+\sin(\theta))} + (1+\sin(\theta)) = 4 - 8\sqrt{|\cos(\theta)|} + 4|\cos(\theta)|.$$

Once again the middle term on the left simplifies, this time to $2|\cos(\theta)|$. Making this substitution, simplifying, and isolating the radical term leads to

$$1 + |\cos(\theta)| = 4\sqrt{|\cos(\theta)|}.$$

Squaring one last time results in

$$1 + 2|\cos(\theta)| + \cos^2(\theta) = 16|\cos(\theta)|.$$

Divide by $|\cos(\theta)|$ and collect the numerical values on the right. Notice that cosine and secant, being reciprocals, always have the same sign, so the final result is

$$\left|\frac{1}{\cos(\theta)}\right| + |\cos(\theta)| = |\sec(\theta) + \cos(\theta)| = \boxed{14}.$$

(HM)

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21. Expand: $\lfloor (x+0.001)^2 - x^2 \rfloor = \lfloor x^2 + 0.002x + 0.000001 - x^2 \rfloor = \lfloor 0.002x + 0.000001 \rfloor$. Thus $2023 \le 0.002x + 0.000001 < 2024$. Since $0.002 = 1/500$ there are $\boxed{500}$ integers that satisfy this inequality. (EA)

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22. Start by taking the base-2 logarithm of each side of the equation: $\log_2(2^{8^x}) = \log_2(8^{2^x})$. Using the laws of logarithms, this becomes $8^x = 2^x \log_2(8) = 3 \cdot 2^x$. Dividing both sides by 2^x yields $4^x = 3$ so $x = \log_4(3)$. The requested ordered pair is $\boxed{(4,3)}$. The restriction placed on a is to prevent equivalent answers such as $x = \log_{16}(9)$. (MF)

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23. This problem could be solved by brute force. There are $8^4 = 4{,}096$ numbers that start with a 1, another 4,096 that start with 2, so after 8,192 numbers the next band of numbers start with 3. The $11{,}923 - 8{,}192 = 3{,}731^{\text{st}}$ of these is needed. Now there are $8^3 = 512$ of these that begin 31, 512 that begin 32, and so on. Note that $7 \times 512 = 3{,}584 < 3{,}731$ and $3{,}731 - 3{,}584 = 147$. Thus, the 147^{th} number that begins with 38 is needed.

Continuing, there are 64 numbers that begin 381, and 64 more that begin 382, which are 128 of those 147 numbers. The 19^{th} number beginning with 383 is needed. There are 8 numbers beginning 3831, 8 more beginning 3832, leaving just three numbers to go... so the desired number is $\boxed{38{,}333}$.

A slightly more clever way to attack the problem is to realize that since only 8 choices are available for each digit of the number, the problem is essentially to convert 11,923 to base 8 (which yields $27{,}223_8$). Then, since the available digits are 1–8 instead of the usual 0–7, each digit of 27,223 should be increased by 1 except the units digit. The units digit should not be increased because the first number on the list, 11,111 corresponds to 00000_8, not 00001_8. So the result is 38,333, as found previously. (SG)

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24. Assume the cube has side length s. Consider the various possibilities for choosing three vertices of a cube.

- If all three vertices belong to the same face of the cube, then their distances from each other will be s, s, and $s\sqrt{2}$.
- If two vertices are at opposite ends of the same edge but the third vertex is not on one of the two faces that have this edge in common, then the distances between the vertices will be s, $s\sqrt{2}$ and $s\sqrt{3}$.
- If no two vertices are on the same edge, then the three of them are the vertices of an equilateral triangle whose sides have length $s\sqrt{2}$.

Three applications of the distance formula show that each of the given points is at distance $\sqrt{98} = 7\sqrt{2}$ from each of the other two. Thus the points are in the third configuration above, with $s = 7$. The surface area of a cube is $6s^2 = 6 \cdot 49 = \boxed{294}$. (MF)

Problem authors:

EA = Former Chicago All-Star Mathlete Eric Auvinen
BB = Former Chicago All-Star Mathlete Brendan Biernacki
MC1 = Chicago ARML Coach Michael Caines
MC2 = Former Chicago All-Star Mathlete Megan Chen
MF = Chicago ARML Coach Micah Fogel
SG = Chicago ARML Coach Steve Goodman
CJ = ARML Contest Writer/Editor (and former Chicago All-Star Mathlete) Chris Jeuell
HM = Former Chicago All-Star Mathlete Holden Mui
JR = Former Chicago All-Star Mathlete Joey Rivkin