2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 1–2 **Name** ______________________________

Time Limit: 10 minutes **School** ______________________________

1. Compute the number of ordered pairs of nonnegative integers (x, y) such that $4^x \cdot 2^{x+y} = 256$.

2. Points C and T lie outside and inside square $ARML$, respectively, so that $\triangle ACR \cong \triangle MTR$, and $\mathrm{m}\angle ACR = 90°$. Suppose that $AC = 1$ and $CT = 24$. Compute $[ARML]$, the area of $ARML$.

ANSWER TO PROBLEM 1

ANSWER TO PROBLEM 2

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 3–4 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

3. In $\triangle ABC$, points M, N, and O lie on $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ respectively, so that $\frac{AM}{MB} = \frac{2}{3}$ and O is the midpoint of $\overline{AC}$. Suppose that $\overline{AN}$, $\overline{CM}$, and $\overline{BO}$ intersect at Q. Compute $\frac{MQ}{QC}$.

4. Compute the number of ways to split a plate of five chocolate-chip cookies and two snickerdoodles among three children, assuming that no child is guaranteed to receive a cookie. (Cookies of each kind are identical and cannot be broken.)

ANSWER TO PROBLEM 3

ANSWER TO PROBLEM 4

Problems 5–6 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

5. Compute the number of values of x, $0 < x < 2\pi$, such that

$$\left(4^{\cos(2x)}\right)^{\sin(2x)} = 2.$$

6. A very confused student mixes up sines, cosines, and logs, and comes up with the "identity" $\log_2(2x) = 1 - 2(\log_2 x)^2$. Although it is not true for all values of x, amazingly, it is true for some values of x. Compute all values of x that satisfy this equation.

ANSWER TO PROBLEM 5

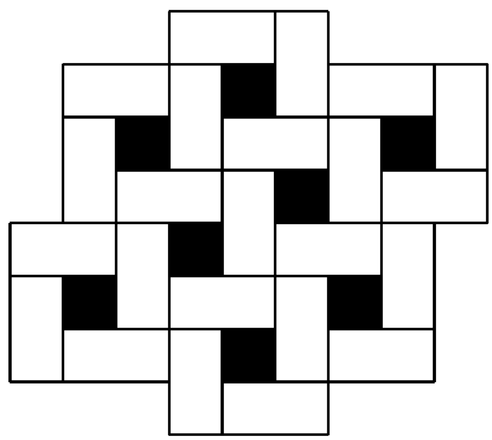
ANSWER TO PROBLEM 6

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 7–8 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

7. A tiling of the plane consists of black 1×1 squares surrounded by white 2×1 rectangles, where the pattern shown below is repeated forever in all directions. Compute the fraction of the plane that is black.



8. Ari, Jonah, and Helen go for a jog around a circular track. Ari and Helen take off in one direction at 6 feet per second and 4 feet per second, respectively, while Jonah runs in the opposite direction at 5 feet per second. All three runners start from the same point at the same time, and the jog finishes when they all three are again at the same point at the same time. Whenever one runner passes another in either direction, they give a high-five. Compute the number of high-fives that occur between the start and the end.

ANSWER TO PROBLEM 7

ANSWER TO PROBLEM 8

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 9–10 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

9. The roots of the polynomial $x^3 + px^2 + 24x + q$ are integers, not necessarily distinct. Suppose that 2 is one of the roots. Compute the greatest possible value of $|q|$.

10. For some integer p, a trapezoid has bases of length 9 and p, and legs of length 9 and 8. Compute the number of possible values of p.

ANSWER TO PROBLEM 9

ANSWER TO PROBLEM 10

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 11–12 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

11. On the complex plane, z and w are numbers satisfying $z^6 = 1$ and $w^4 = -1$. Given that 0, z, w, and $z + w$ form a quadrilateral with nonzero area, the minimum possible area of the quadrilateral can be expressed as $\frac{\sqrt{a} - \sqrt{b}}{c}$, where a, b, and c are positive integers, and a and b are squarefree. Compute $a + b + c$.

12. Rene and Blaise play a game flipping a weighted coin that comes up heads with probability p. Rene wins if the coin comes up heads twice in a row, while Blaise wins if the coin comes up tails twice in a row; if the flips are different, they continue flipping until two consecutive flips are the same. After a while, nobody has won and the two get bored with the game, so after flipping one last tails, they agree to split the pot evenly. Assuming this division is fair, compute p.

ANSWER TO PROBLEM 11

ANSWER TO PROBLEM 12

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 13–14 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

13. In convex quadrilateral $ARML$, $AR = 6$, $RM = 8$, and $\overline{AR} \perp \overline{RM}$. Let $m\angle ALM = 120^\circ$. Compute the maximum possible area of $ARML$.

14. The *base-factorial* representation of a real number $0 \le r < 1$ is the number $0.b_1b_2b_3\ldots$, where:

- for every positive integer i, b_i is a nonnegative integer less than or equal to i, and
- $\dfrac{b_1}{2!} + \dfrac{b_2}{3!} + \dfrac{b_3}{4!} + \cdots = r$.

Compute the base-factorial representation of $\dfrac{20}{21}$.

ANSWER TO PROBLEM 13

ANSWER TO PROBLEM 14

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 15–16 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

15. Let $\alpha = \cot\left(\frac{\pi}{12}\right)$ and let $\beta = \tan\left(\frac{\pi}{12}\right)$. Compute $\lfloor \alpha^{12} + \beta^{12} \rfloor$.

16. An equilateral triangle ABC in three-dimensional space can be *spiked* by situating a point P in space so that triangles PAB, PBC, and PAC are isoseceles right triangles with right angles at P. Each face of a regular tetrahedron is spiked, using a point located outside the tetrahedron. The resulting figure is a convex polyhedron. Given that the edges of the tetrahedron have length 12, compute the volume of its spiked counterpart.

ANSWER TO PROBLEM 15

ANSWER TO PROBLEM 16

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 17–18 **Name** ______________________________

Time Limit: 12 minutes **School** ______________________________

17. Compute the sum of all values of x such that $\log_2(\log_4 x) + \log_4(\log_2 x) = \dfrac{7}{2}$.

18. Complete the cross-number puzzle below, where each Across answer is a four-digit number and each Down answer is a three-digit number. No answer begins with the digit 0.
Your answer must be written in the space at the bottom of this page, not the grid to the right!

Across
1. Just like the number 2022, all but one of its digits are the same
5. All digits are distinct primes
6. No digit is a multiple of 3

Down
1. A perfect cube
2. The rightmost three digits of the least multiple of 2022, none of whose digits is 0
3. A Fibonacci number
4. A palindrome

1	2	3	4
5			
6			

ANSWER TO PROBLEM 17

ANSWER TO PROBLEM 18

1	2	3	4
5			
6			

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 19–20 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

19. For $x > 0$, compute the minimum possible value of $4096^{x-\sqrt{x}}$.

20. From left to right, the last eight digits of $n!$, expressed in base 10, are 20,000,000. Compute n.

ANSWER TO PROBLEM 19

ANSWER TO PROBLEM 20

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 21–22 **Name** ____________________________

Time Limit: 10 minutes **School** ____________________________

21. Benny the bug starts at point A and travels 1 meter in a straight line. Benny then turns left $90°$ and travels another 1 meter in a straight line. Benny turns $90°$ left and travels 2 meters in a straight line, turns left again, and travels another 2 meters in a straight line. Benny continues in this fashion, adding 1 meter to the length of his walks every two segments and turning left $90°$ after each segment. After Benny has walked 2022 meters, compute the number of meters he is from his starting point.

22. Regular nonagon $MATHISFUN$ has side length 17. Compute $IF \cdot IT - IS \cdot NT$.

ANSWER TO PROBLEM 21

ANSWER TO PROBLEM 22

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Problems 23–24 **Name** ______________________________

Time Limit: 10 minutes **School** ______________________________

23. In a typical recent year, the gap between the amount of federal income taxes owed and taxes actually paid is about \$441,000,000,000, according to the IRS. Compute the sum of the distinct prime factors of 441,000,000,000.

24. Segment $\overline{DK}$ has length $\frac{40}{7}$. Points Y and P lie on $\overrightarrow{DK}$, with Y between D and K, and K between D and P, so that $\frac{YD}{YK} = \frac{DP}{KP}$. Let y be the greater of DY and DP, and let p be the lesser of DY and DP. Given that $yp = 40$, compute the ordered pair (y, p).

ANSWER TO PROBLEM 23

ANSWER TO PROBLEM 24

2022 CHICAGO AREA ALL-STAR MATH TEAM TRYOUTS

Part I Answers

1. 3
2. 289
3. $\frac{3}{5}$ (or 0.6)
4. 126
5. 4
6. $1, \frac{\sqrt{2}}{2}$ (must have both; $2^{-1/2}$ is acceptable)
7. $\frac{1}{5}$ (or 0.2)
8. 19
9. 180
10. 22
11. 12
12. $\frac{\sqrt{5}-1}{2}$
13. $\frac{72+25\sqrt{3}}{3}$
14. 0.122415

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Part II Answers

15. 7300802

16. $432\sqrt{2}$

17. 256

18.

1	1	3	1
2	3	7	5
5	2	7	1

19. $\frac{1}{8}$ (or 0.125)

20. 34

21. $2\sqrt{221}$

22. 289

23. 17

24. $(10, 4)$

Tryout Results 2022

Item Analysis

($n = 186$ for Q1–Q24)

Q	# right	%
1	119	64%
2	123	66%
3	125	67%
4	83	45%
5	89	48%
6	72	39%
7	125	67%
8	166	89%
9	154	83%
10	40	22%
11	113	61%
12	40	22%

Q	# right	%
13	125	67%
14	135	73%
15	80	43%
16	28	15%
17	110	59%
18	59	32%
19	47	25%
20	22	12%
21	49	26%
22	128	69%
23	121	65%
24	98	53%

Distribution of Top Scores

Score	n
24	2
23	5
22	4
21	7
20	5
19	6
18	9
17	11
16	12
15	11
14	16