Chicago Area All-Star Math Team

2022 Tryout Solutions

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1. Use the laws of exponents to rewrite the given equation: $4^x \cdot 2^{x+y} = 2^{2x} \cdot 2^{x+y} = 2^{3x+y}$. On the right-hand side $256 = 2^8$. So $3x + y = 8$ and x and y must be nonnegative integers. Each choice of x leads to a unique y. In fact, the solutions are $(x, y) = (0, 8)$, $(1, 5)$, and $(2, 2)$. Larger choices of x lead to negative y. So there are $\boxed{3}$ such ordered pairs.

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2. Since $\angle CRM - \angle CRA$ leaves right angle $\angle ARM$, and it is given that angles $\angle CRA$ and $\angle TRM$ are congruent, it must be the case that $\angle CRM - \angle TRM = \angle TRC$ is also a right angle. It is also given that $CR = TR$ so $\triangle CRT$ is an isosceles right triangle whose hypotenuse is given to be 24. That means that the legs are $CR = TR = 12\sqrt{2}$. Turning to right triangle $\triangle ACR$ whose legs are 1 and $12\sqrt{2}$, the hypotenuse is $AR^2 = 1^2 + (12\sqrt{2})^2 = 289$. But note that the area requested is simply $AR^2 = \boxed{289}$!

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3. There are a number of ways to solve this problem. Perhaps the most direct is to apply Menelaus's Theorem to $\triangle AMC$ with transversal $\overline{BO}$ leads to $\frac{AB}{BM} \cdot \frac{MQ}{QC} \cdot \frac{CO}{OA} = 1$. The ratio $\frac{AM}{MB} = \frac{2}{3}$ so $\frac{AB}{BM} = \frac{AM+MB}{BM} = \frac{2}{3} + 1 = \frac{5}{3}$. Since O is given to be the midpoint of $\overline{AC}$ the ratio $\frac{CO}{OA} = 1$. Thus $\frac{MQ}{QC} = \boxed{\frac{3}{5}}$.

A more brute force approach involves choosing lengths for the sides. Since there are 2's and 3's in the problem, and $2 + 3 = 5$, and the numbers 3 and 5 remind us of the 3-4-5 triangle, choose lengths $AM = 2$, $MB = 3$, and $BC = 5$. Then $MC = 4$. Applying Pythagoras to $\triangle AMC$ yields $AC = \sqrt{20}$. Now $\triangle QOC$ is also a right triangle similar to triangle $\triangle AMC$ because they share the angle at C. Since $OC = \sqrt{20}/2 = \sqrt{5}$, by similarity the length $QC = \frac{\sqrt{20}}{4} \cdot \sqrt{5} = \frac{5}{2}$. Then $QM = 4 - \frac{5}{2} = \frac{3}{2}$ and $\frac{MQ}{QC} = \boxed{\frac{3}{5}}$.

Finally, for those who love mass points, place a mass of 3 at A, a mass of 2 at B, and a mass of 3 at C. These choices cause the triangle to balance along segments $\overline{MC}$ and $\overline{BO}$. So the triangle will balance at point Q. Now slide the two masses on side $\overline{AB}$ to M to find that $\overline{MC}$ balances at Q with a mass of 5 at M and a mass of 3 at C, and once again the ratio $\frac{MQ}{QC} = \boxed{\frac{3}{5}}$.

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4. The distributions of the chocolate-chip cookies and the snickerdoodles are independent of each other. That means the total number of possible distributions can be determined by finding the number of ways to distribute each cookie and multiplying.

There are six ways to distribute the snickerdoodles. Either both are given to one child (three choices of which child) or one is given to each of two children (three choices of which child does not get a cookie).

To split up the chocolate-chip cookies, consider all possible distributions. They could be split 5-0-0 (three ways to do this), 4-1-0 (six ways—three choices of who gets 4 and then two choices of who gets 1), 3-2-0 (six ways), 3-1-1 (three ways), or 2-2-1 (three ways). That adds up to 21 possible distributions of the chocolate-chip cookies.

That makes the final total $6 \cdot 21 = \boxed{126}$.

An important technique for mathletes is to learn how to count the number of ways to put k identical balls into n distinguishable urns. In this case, 5 cookies (balls) distributed among 3 children (urns). The technique is known to some as *sticks-and-stones* because it can be explained by taking k stones representing the balls and $n-1$ sticks and seeing how many ways there are to place the sticks among the stones. The sticks act as dividers for which urn the stones are going into. For instance, the arrangement $**|**|*$ (where $*$ is a stone and $|$ is a stick) represents the first two urns getting two balls each and the last urn getting one ball. The arrangement $***||**$ represents three balls in the first urn and two in the last, with none in the middle. The arrangement $|*****|$ represents the middle urn getting all the balls. Since there are $k+n-1$ total symbols and $k-1$ are sticks, the number of choices is $\binom{k+n-1}{k-1}$. In the case of the chocolate-chip cookies, $k=5$ and $n=3$ so there are $\binom{7}{2} = 21$ arrangements. For the snickerdoodles there are $\binom{2+3-1}{3-1} = \binom{4}{2} = 6$ arrangements.

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5. First, use the laws of exponents to rewrite the given equation as $4^{(\cos(2x)\sin(2x))} = 2$. Now $\cos(2x)\sin(2x) = \frac{1}{2}\sin(4x)$ by the double-angle formula. Since $2 = 4^{1/2}$ the equation is satisfied when $\sin(4x) = 1$. This happens $\boxed{4}$ times in the interval $0 < x < 2\pi$.

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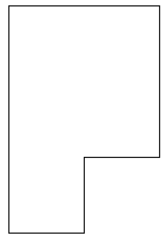
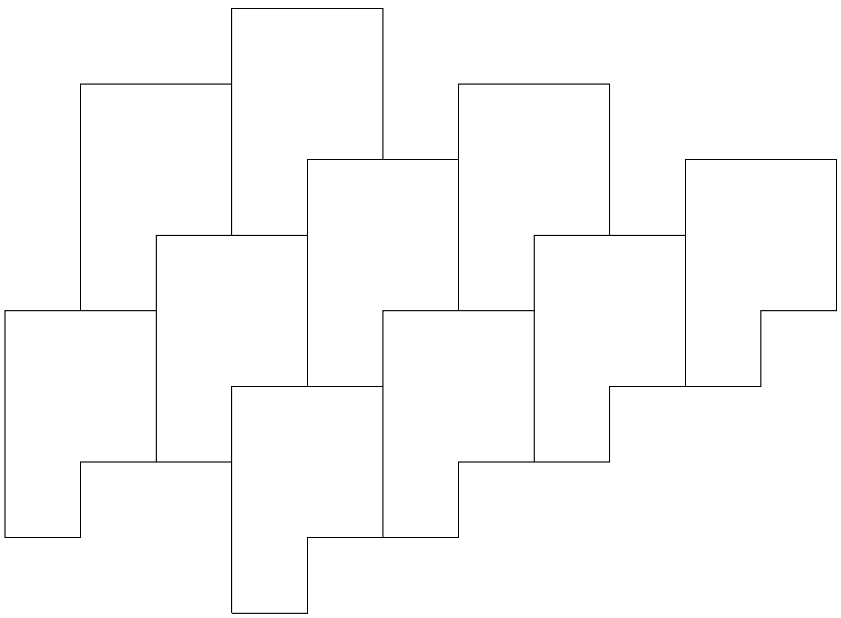
6. Apply the laws of logarithms to the left-hand side to obtain $\log_2(2) + \log_2(x) = 1 + \log_2(x)$. Setting this equal to $1 - 2(\log_2(x))^2$ allows the 1's to cancel, leaving $\log_2(x) = -2(\log_2(x))^2$. Letting $y = \log_2(x)$ transforms the equation to $y = -2y^2$ which has solutions $y = 0$ and $y = -1/2$. So $\log_2(x) = 0$ or $\log_2(x) = -1/2$, giving $x = \boxed{1 \text{ or } 2^{-1/2}}$.

7. Take a single black square and the rectangles that border it above and to the left, and fuse them together into a hexagon, as shown below:



These hexagons act as tiles (the mathematical term is *fundamental regions*) and the entire plane can be made from them as shown:



Since each fundamental region has area 5 and the black area is 1, $\boxed{\frac{1}{5}}$ of the plane is black.

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8. Consider the situation from Jonah's point of view. Jonah could pretend to be jogging in place while the track is rotating in the opposite direction at 5 feet per second. The other two runners are traveling along the moving track at 6 and 4 feet per second, so their speeds *relative to Jonah* are $5 + 6 = 11$ and $5 + 4 = 9$ feet per second, respectively. For all three runners to meet, both Ari and Helen must have run an integral number of times around the track relative to Jonah. Since Ari runs 11/9 times as fast as Helen, the first time this occurs is when Ari has run 11 laps and Helen has run 9. Since Ari has run 11 laps, he has passed Jonah 10 times (stopping at the end of the 11th) so delivered 10 high-fives. Similarly, Helen has given 8 high-fives to Jonah. And since Ari has run around the track two more times than Helen, Ari must have passed Helen once (one more high-five). So the total number of high-fives between all the runners is $10 + 8 + 1 = \boxed{19}$.

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9. Let the roots be 2, a, and b. Multiplying out the factored polynomial $(x-2)(x-a)(x-b)$ yields $x^3 + (-2 - a - b)x^2 + (2a + 2b + ab)x - 2ab$. So $ab + 2a + 2b = 24$. Factor the left-hand side of this equation: $(a + 2)(b + 2) - 4 = 24$ so $(a + 2)(b + 2) = 28$. Since a and b are integers, it must be that $a + 2$ and $b + 2$ are factors of 28. For instance, $a + 2 = 4$ and $b + 2 = 7$ leads to $(a, b) = (2, 5)$. The other possibilities for (a, b) are $(0, 12)$, $(-1, 26)$, $(-3, -30)$, $(-4, -16)$, and $(-6, -9)$ or their reverses $(12, 0)$, $(26, -1)$, etc. The greatest that $|q| = |2ab|$ can be from these possibilities is $|q| = |2(-3)(-30)| = \boxed{180}$.

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10. Let h be the height of the trapezoid. This height must be at most 8, the length of one of the legs of the trapezoid. If both angles between the base of length 9 and the two legs are acute, then the opposite base has length $9 - \sqrt{9^2 - h^2} - \sqrt{8^2 - h^2}$ by using the Pythagorean Theorem twice. If h is too small, this value would be negative; it is zero when h is the altitude to one of the sides of length 9 in the 9-9-8 triangle. As h increases from this value up to $h = 8$ the length of the opposite base increases from 0 to $9 - \sqrt{17}$. Thus, if both base angles are acute and the opposite base is required to be an integer, the only possible lengths are 1, 2, 3, and 4.

If the angle to the leg of length 8 is obtuse then $0 < h < 8$ and the opposite side now has length $9 - \sqrt{9^2 - h^2} + \sqrt{8^2 - h^2}$ which varies in the range $(9 - \sqrt{17}, 8)$. Thus, p can attain the values 5, 6, or 7.

If the angle between the base of length 9 and the leg of length 9 is obtuse, then the length of the opposite base is $9 + \sqrt{9^2 - h^2} \pm \sqrt{8^2 - h^2}$, with the plus chosen if the other base angle to the leg of length 8 is also obtuse, the minus chosen if acute. Again, h may vary in the range $0 < h < 8$. The smallest this length could be is if h is near zero and the minus sign is chosen, making the opposite base just larger than 10; the largest it be is when h is near zero and the plus sign is chosen, making the opposite base just smaller than 26. All numbers in between are possible values, so p can take on values between 11 and 25 inclusive. Thus, altogether there are $15 + 7 = \boxed{22}$ possible values for p.

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11. An application of de Moivre's Theorem shows that z must be a complex number of the form $1\operatorname{cis}(\theta)$ where θ is a multiple of $60°$. Another application shows that w is of the form $1\operatorname{cis}(\omega)$ where ω is an odd multiple of $45°$. Sketching the numbers in an Argand diagram shows that 0, z, w, and $z + w$ form a parallelogram whose area is seen to be $1 \cdot 1 \cdot |\sin(\theta - \omega)|$. Since the area is nonzero and different choices of θ and ω can only make $\theta - \omega$ be a multiple of $15°$, the smallest $|\sin(\theta)|$ can be is $\sin(15°)$. Many mathletes know this value, but if you don't you can quickly compute it using angle addition formulas by finding $\sin(60° - 45°) = \sin(60°)\cos(45°) - \cos(60°)\sin(45°)$ which evaluates to $\frac{\sqrt{6}-\sqrt{2}}{4}$.

Thus $a = 6$, $b = 2$, and $c = 4$, so $a + b + c = \boxed{12}$.

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12. Historically, the field of probability was created to determine how to split pots in games like this that were left unfinished. A fair division of the pot is one where each contestant receives a fraction of the pot that is equal to his or her probability of winning the pot were the game to be completed. In this case, the value of p must be determined so that the probability of each player winning if the game were continued, and given that tails has just been tossed, is equal to the share of the pot they are taking, which is 1/2.

To that end, define B_t to be the probability that Blaise would win the continued game given that the last throw was tails, with B_h, R_t, and R_h defined similarly for the two players and the two possible last throws. These values can be computed using the following information:

- Blaise wins if the next throw is tails (probability $(1-p)$) or if it is heads (probability p) and he wins the ensuing game starting with heads: $B_t = (1-p) + pB_h$.
- Had the previous throw been heads, Blaise can only win if the next throw is tails *and* he wins the subsequent game starting from tails: $B_h = (1-p)B_t$.
- Similarly, Rene wins starting from heads if the next throw is heads, or if it is tails and he wins the subsequent game starting from tails: $R_h = p + (1-p)R_t$.
- Finally, Rene can win starting from tails is the next throw is heads and then he wins the subsequent game starting from heads: $R_t = pR_h$.

Concentrate on Blaise's probability of winning. Using the first two equations above, substitute the second into the first to find $B_t = (1-p) + p(1-p)B_t$. This can be solved for B_t to obtain $B_t = \frac{1-p}{1-p+p^2}$. Since the probability that Blaise wins should be 1/2, set these equal:

$$\begin{aligned}\frac{1-p}{1-p+p^2} &= \frac{1}{2}\\ 2-2p &= 1-p+p^2\\ p^2+p-1 &= 0.\end{aligned}$$

This can be solved using the quadratic formula, $\frac{-1\pm\sqrt{5}}{2}$ and the probability must be positive, so $p = \boxed{\dfrac{-1+\sqrt{5}}{2}}$.

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13. Diagonal $\overline{AM}$ divides the quadrilateral into two triangles, $\triangle ARM$ and $\triangle ALM$. Since $\overline{AR} \perp \overline{RM}$, the triangle $\triangle ARM$ is a 6-8-10 right triangle with right angle at R; its area is $\frac{6\cdot 8}{2} = 24$. The maximum area of the quadrilateral occurs when the area of $\triangle ALM$ is maximized, which happens when it is isosceles. In this case, the base angles will be 30°

so the altitude will be $5\tan(30°) = 5\sqrt{3}/3$. Since the base is $AM = 10$, the area of $\triangle ALM$ will be $25\sqrt{3}/3$. That makes the total area of the quadrilateral $\boxed{24 + 25\frac{\sqrt{3}}{3}}$.

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14. **Note:** the problem as stated originally had an error in it, requiring $b_i < i$. This rendered the problem impossible as stated. The solution that follows is for the corrected version of the problem, where $b_i \le i$.

The usual method of converting fractions to and from different base representations is to repeatedly multiply by the base, and the next digit in the fraction's expansion is the whole part of the result. Apply the same method, except instead of multiplying by the base each time, multiply by 2, then by 3, then by 4, and so on. The results are:

$$\begin{aligned} \frac{20}{21} &= \frac{b_1}{2!} + \frac{b_2}{3!} + \frac{b_3}{4!} + \cdots \\ 2 \cdot \frac{20}{21} &= b_1 + \frac{2b_2}{3!} + \frac{2b_3}{4!} + \cdots \\ \frac{40}{21} = 1\,\frac{19}{21} &= b_1 + \frac{2b_2}{3!} + \frac{2b_3}{4!} + \cdots \end{aligned}$$

So b_1 must be the whole part of the left-hand side, which is 1. Then

$$\begin{aligned} \frac{19}{21} &= \frac{2b_2}{3!} + \frac{2b_3}{4!} + \cdots \\ 3 \cdot \frac{19}{21} &= \frac{3!b_2}{3!} + \frac{3!b_3}{4!} + \cdots \\ \frac{57}{21} = 2\,\frac{5}{7} &= b_2 + \frac{3!b_3}{4!} + \frac{3!b_4}{5!} + \cdots \end{aligned}$$

Now, b_2 must be the whole part of the left-hand side, which is 2. Next we multiply both sides by 4 to obtain $2\frac{6}{7} = b_3 + \frac{4!b_4}{5!} + \cdots$ to find $b_3 = 2$. Continue in this fashion, multiplying by 5, 6, 7, ... After multiplying by 7, the remaining fractional part is zero, so all further multiplications would simply produce zeroes. We can stop the expansion at thie time. The final result is that in base-factorial notation, $\frac{20}{21} = \boxed{0.122415}$.

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15. First, use the tangent half-angle formula $\tan(\theta/2) = \frac{1-\cos(\theta)}{\sin(\theta)}$ to find that $\tan(\pi/12) = \frac{1-\cos(\pi/6)}{\sin(\pi/6)} = \frac{1-\sqrt{3}/2}{1/2} = 2 - \sqrt{3}$. Then $\cot(\pi/12) = \frac{1}{\tan(\pi/12)} = 2 + \sqrt{3}$.

Now here is a neat trick for finding $c_n = a^n + \dfrac{1}{a^n}$. Note that $c_n c_1 = a^{n+1} + a^{n-1} + \dfrac{1}{a^{n-1}} + \dfrac{1}{a^{n+1}} = c_{n+1} + c_{n-1}$. So we can set up a recurrence

$$c_n = \begin{cases} a^0 + \frac{1}{a^0} = 1 + 1 = 2 & n = 0 \\ a + \frac{1}{a} & n = 1 \\ c_1 c_{n-1} - c_{n-2} & n \ge 2 \end{cases}.$$

In the current problem, $c_1 = (2+\sqrt{3}) + (2-\sqrt{3}) = 4$. So we start our recurrence:

$$\begin{aligned} c_0 &= 2 \\ c_1 &= 4 \\ c_2 &= 4c_1 - c_0 = 16 - 2 = 14 \\ c_3 &= 4c_2 - c_1 = 56 - 4 = 52 \\ &\vdots \\ c_{12} &= 4c_{11} - c_{10} = 4 \cdot 1{,}956{,}244 - 524{,}174 = \boxed{7{,}300{,}802}. \end{aligned}$$

Since this is already an integer, the floor can be ignored, making this the final answer.

For those who like efficiency, once c_2 has been found, restart the recurrence from c_0 and c_2 and the next step will be c_4 instead of c_3. Then starting the recurrence again with c_0 and c_4 will get c_8 and then c_{12}, so the answer can be computed in four steps rather than 12.

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16. One method for solving this problem would be to find (or remember) the formula for the volume of a regular tetrahedron, determine the volume of the "spikes" that are added to each side, and add to find the total volume. There is a cleverer solution available, however!

There is a very convenient regular tetrahedron with vertices at $(0,0,0)$, $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$. Using this as the base tetrahedron it is not too hard to convince oneself that spiking this tetrahedron results in the unit cube! So to solve the problem all that needs be done is to scale things up to the right size.

The sides of the convenient tetrahedron are easily seen to have length $\sqrt{2}$. Thus the sides of the tetrahedron in the problem are $6\sqrt{2}$ times as long. That means all relevant volumes are $(6\sqrt{2})^3 = 432\sqrt{2}$ times as much. Since the volume of the unit cube is 1, the volume of the final spiked tetrahedron (cube!) in the problem is $\boxed{432\sqrt{2}}$.

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17. Since the left-hand side is an increasing function of x there is only one possible solution. To find it, choose a clever expression to represent x. For example, let $x = 2^{4^y}$. Then $\log_4(\log_2(x)) = \log_4(4^y) = y$ while $\log_2(\log_4(x)) = \log_2(4^y \log_4(2)) = \log_2(\frac{1}{2}4^y) = -1 + y\log_2(4) = 2y - 1$. Thus, the equation becomes $y + 2y - 1 = \frac{7}{2}$ so $y = \frac{3}{2}$. Substituting, $x = 2^{4^{3/2}} = 2^8 = \boxed{256}$.

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18. For 1-Down, the only 3-digit cubes are 125, 216, 343, 512, and 729. But since neither 1 nor 4 is prime (violating 5-Across) and 9 is a multiple of 3 (violating 6-Across), 1-Down must be 125.

For 2-Down, the least multiples of 2022 are 2022, 4044, 6066, 8088, 10110, and 12132 which is the least to have no 0 digits. The rightmost three digits of this number are 132.

For 3-Down, the three-digit Fibonacci numbers are 144, 233, 377, 610, and 987. But 5-Across rules out 144, 610, and 987, while 6-Across rules out 233 and 610. So this clue must have answer 377.

Now 1-across reads 113? and must have all but one its digits the same. That requires the unknown digit to be a 1.

5-Across has already used the primes 2, 3, and 7, so the final digit must be a 5.

Lastly, 4-Down's last digit is the same as its first: 1.

With these numbers in place, the completed puzzle is

1	1	3	1
2	3	7	5
5	2	7	1

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19. This problem hinges on finding the minimum possible value of the exponent $x - \sqrt{x}$. But since x is positive, let $x = y^2$. Then the minimum value of $y^2 - y$ must be found. But finding the minimum value of the quadratic $ay^2 + by + c$ should be child's play for a mathlete; it occurs at the vertex $y = -b/2a$. Here, $y = 1/2$ and $y^2 - y = -1/4$. So the answer to the question is $4096^{-1/4} = (2^{12})^{-1/4} = 2^{-3} = \boxed{1/8}$.

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20. To find the number of trailing zeroes of $n!$, repeatedly divide n by 5, adding the quotients and discarding the remainders. For instance, $58 \div 5 = 11$ (ignoring the remainder of 3), $11 \div 5 = 2$, and $2 \div 5 = 0$. Adding the quotients gives $11 + 2 = 13$ so 58! has 13 trailing zeroes. For this problem, the factorial has seven trailing zeroes. Some quick work shows that n must lie between 30 and 34 inclusive.

There is an algorithm for finding the rightmost non-zero digit of $n!$ but it is rather tedious and it will be faster to just compute the digit directly for the small numbers in this problem. Note that 30! has seven factors of 5, so to find its rightmost non-zero digit multiply together all the units digits (ignoring trailing zeroes) and dividing by seven 2's (done below by dividing 2, 4, 6, and 8 by all their 2's) of the numbers up to 30: (the numbers that have been divided by 2 or 5 are boldfaced)

$$1 \cdot \mathbf{1} \cdot 3 \cdot \mathbf{1} \cdot \mathbf{1} \cdot \mathbf{3} \cdot 7 \cdot \mathbf{1} \cdot 9 \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot \mathbf{3} \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot \mathbf{4} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot \mathbf{1} \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot \mathbf{3}$$

whose units digit is 8. Multiplying by 31 also gives a rightmost non-zero digit of 8. By 32, a 6. By 33, 8 again. Finally, multiplying this by $\boxed{34}$ yields the desired 2.

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21. After $2k$ legs of his journey, Benny has traveled $1 + 1 + 2 + 2 + \cdots + k + k = k(k+1)$ meters. The first time this is at least 2022 is for $k = 45$ when Benny would have traveled $45 \cdot 46 = 2070$ meters. He will have to back up 48 meters from where he would end up at this time! But where exactly is that? By placing point A at the origin and having Benny first walk in the direction of the positive x-axis, and tracing out the first few steps of Benny's journey, his location after 2, 4, 6, 8, 10, ... steps is $(1, 1)$, $(-1, -1)$,

$(2,2)$, $(-2,-2)$, $(3,3)$, $\ldots$. Since $k = 45$ and Benny has traveled $2k$ legs of his walk, he would be at $(23, 23)$. But he must back up 48 meters from that point. His last trip of 45 meters came from $(23, -22)$. Back up 3 more meters and he was at $(20, -22)$. This is $\sqrt{20^2 + 22^2} = \sqrt{884} = \boxed{2\sqrt{221}}$ meters from his starting point.

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22. Because the polygon is regular, $IT = IF = HS$ and $NT = HF$. Then the requested quantity $IF \cdot IT - IS \cdot NT$ is equal to $IF \cdot HS - IS \cdot HF$. The reason for using these lengths instead of the originals is because all of these are distances between vertices of quadrilateral $HISF$, which is cyclic. Applying Ptolemy's Theorem gives $IF \cdot HS - IS \cdot HF = HI \cdot SF = 17 \cdot 17 = \boxed{289}$.

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23. Removing all the factors of 2 and 5 leaves $441 = 21^2$ whose prime factors are 3 and 7. Adding: $2 + 3 + 5 + 7 = \boxed{17}$.

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24. The given equality of ratios of lengths can be rewritten as $\dfrac{y}{\frac{40}{7} - y} = \dfrac{p}{p - \frac{40}{7}}$. Cross-multiplying leads to $yp - \frac{40}{7}y = \frac{40}{7}p - yp$. It is given that $yp = 40$ so this simplifies to $\frac{40}{7}(p + y) = 2yp = 80$, thus $p + y = 14$.

Now $(p + y)^2 = 14^2 = 196$. But $(p + y)^2 = p^2 + 2yp + y^2$. Subtracting $4yp$ from this quantity reveals that $p^2 - 2yp + y^2 = 196 - 160 = 36$. The left-hand side of this is $(p - y)^2$. So $p - y = \pm 6$. Since it is given that $y > p$, it must be that $p - y = -6$. Together with $p + y = 14$, the pair (y, p) is now easily determined to be $\boxed{(10, 4)}$.