CHICAGO ARML

Chicago Area All-Star Math Team

2020 Tryout Solutions

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1. While you can certainly brute force the arithmetic, there is a neat pattern lurking underneath. Start with some easier test cases: $1 \cdot 2 \cdot 3 \cdot 4 + 1 = 25 = 5^2$, $2 \cdot 3 \cdot 4 \cdot 5 + 1 = 121 = 11^2$, and if you're patient, $3 \cdot 4 \cdot 5 \cdot 6 + 1 = 361 = 19^2$. It appears you get the square of one more than the product of the least and greatest numbers in the product. That is, if you have $\sqrt{n(n+1)(n+2)(n+3)+1}$ the result is $n(n+3)+1$. Is this true?

The combination $n(n+1)(n+2)(n+3)+1$ can be regrouped to yield

$$\Big(n(n+3)\Big)\Big((n+1)(n+2)\Big) + 1 = (n^2+3n)(n^2+3n+2)+1.$$

Let $q = n^2 + 3n$. Then we have $q(q+2)+1 = q^2+2q+1 = (q+1)^2$. So the combination under the radical is always the square of n^2+3n+1 as we surmised in the paragraph above.

In this case, $\sqrt{(17)(18)(19)(20)+1} = \sqrt{(17 \cdot 20 + 1)^2} = \boxed{341}$.

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2. The table below lists the possible combinations of values shown on the dice that add to 11, the number ways to obtain that combination, and whether or not that combination contains any prime values. Note that there are six ways to obtain a combination of three different numbers (for example, 6-4-1, 6-1-4, 4-6-1, 4-1-6, 1-6-4, and 1-4-6) and three ways to obtain a combination that contains two different numbers (5-5-1, 5-1-5, 1-5-5).

Combination	# of ways	Contains a prime?
6-4-1	6	No
6-3-2	6	Yes
5-5-1	3	Yes
5-4-2	6	Yes
5-3-3	3	Yes
4-4-3	3	Yes

Totaling the middle column, there are 27 ways to roll a total of 11, but only 6 of them do not contain a prime value. So the requested probability is $6/27 = \boxed{2/9}$.

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3. Since the bottom half of the volume is full, the top half of the volume will be empty. The top half is itself an equilateral triangular prism with the same 20 cm length as the whole prism. So all that is needed is to find the length of a side of an equilateral triangle whose area is half that of an equilateral triangle with sides of length 10 cm. Area scales as the square of the side length, so to halve the area, we divide the side's length by $\sqrt{2}$. Thus the water's top surface will be a rectangle of length 20 cm and width $10/\sqrt{2}$ cm. Multiplying yields an area of $\boxed{100\sqrt{2}}$ cm^2.

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4. The sum of the roots of a degree-d polynomial whose leading coefficient is 1 is the opposite of the coefficient of the degree-$(d-1)$ term. So we expand the polynomials to obtain the degree-2020 and 2019 terms. The result is

$$\begin{aligned} &x^{2020} - 2020 \cdot 1x^{2019} + \text{(lower order terms)} \\ &+ x^{2020} + 2020 \cdot 2x^{2019} + \text{(lower order terms)} \\ &+ x^{2020} - 2020 \cdot 3x^{2019} + \text{(lower order terms)} \\ &\vdots \\ &+ x^{2020} - 2020 \cdot 2019x^{2019} + \text{(lower order terms)} \\ &+ x^{2020} + 2020 \cdot 2020x^{2019} + \text{(lower order terms)} \end{aligned}$$

Notice that in the second column above, the terms alternate $-1+2$, then $-3+4$, and so forth, each pair adding to 1. There are 1010 pairs. So adding all these expansions, we find that the original polynomial is $2020x^{2020} + 2020 \cdot 1010x^{2019} +$ (lower order terms). Factoring out 2020 to obtain a polynomial with leading coefficient 1, the opposite of the coefficient of the x^{2019} term is $\boxed{-1010}$.

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5. From the angle bisector theorem, $LV/VH = LS/HS$. Substituting the known values and clearing the fractions, $VH \cdot LS = 70$. Since VH and LS are integers, the pair with the least sum whose product is 70 is 7 and 10 (either could be VH with the other being LS). These numbers fit with the triangle inequality, so the triangle exists, and the perimeter will be $14 + 10 + 5 + 7 = \boxed{36}$.

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6. Temporarily assume the word could only be spelled clockwise. Starting from the R, the probability that the next letter clockwise is an E is $2/6 = 1/3$. Then a T must occur, which would occur with probability $2/5$ (if we had the E second, then there are only five letters remaining, two of which are T's). Then we must get the other E; probability $1/4$. Then $2/3$ to get an S and $1/2$ to get the last T and $1/1$ to get the final S. The multiplication rule implies the probability will be $\frac{2}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{8}{720} = \frac{1}{90}$. But since we can actually proceed either clockwise or counterclockwise there are twice as many arrangements that spell *RETESTS* correctly, so our final probability is $\boxed{1/45}$.

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7. Because of the absolute values, the region is symmetric with respect to the y-axis. So consider the portion of the region in the first quadrant, and multiply the resulting area by 2.

In the first quadrant, the region is bounded by the y-axis and the lines $x + y \geq k$ and $2x + y \leq 2k$. This is a triangle with vertices $(k, 0)$, $(0, k)$, and $(0, 2k)$. Using the base as the segment along the y-axis, this triangle has base and height k, so its area is $k^2/2$.

Multiplying by 2, we need $k^2 = 34$ so $k = \boxed{\sqrt{34}}$.

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8. A necessary condition for a system of linear equations to have no solutions is for the determinant of the matrix of coefficients to be 0. Expanding along the bottom row, the determinant of the matrix is $-47a + 94 \cdot 17 - 47b = 47(34 - (a + b))$. Clearly, this is 0 when $a + b = \boxed{34}$.

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9. Let the dimensions of P be $x \times y$. Then the outer dimensions of the walkway around P are $(x + 2w) \times (y + 2w)$. So the area of the walkway around P is $(x + 2w)(y + 2w) - xy$ $= 2w(x + y) + 4w^2$. Similarly, the area of the walkway around Y is $2w(x + 6 + y + 6) + 4w^2$. Subtracting, we learn that the difference is $24w$. This is given to be 150, so $w = 150/24 = \boxed{25/4}$.

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10. Since the sum of the m numbers in A is $2m$, their average must be $2m/m = 2$. Thus the numbers in A are symmetric around the number 2, so they must be the integers from $2 - (m - 1)/2$ up to $2 + (m - 1)/2$. The greatest element of A is thus $2 + (m - 1)/2$.

Similarly, the average of the numbers in B is $1/2$, so B is symmetric with respect to $1/2$. Thus B consists of the numbers from $1 - m$ up to m, and the greatest element of B is m.

The difference is between the greatest elements of A and B is $3/2 - m/2$. Since m is not negative, the only solution that makes the absolute value of this difference equal to 17 is $m = \boxed{37}$.

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11. For all four sums to be odd, each tile-taker must grab one even and one odd tile. Once Xander takes his first tile, there are four tiles of the opposite parity and three of

the same parity remaining in the bag, so the probability that his second tile is the opposite parity is 4/7. Similarly, given that Xander succeeded in taking one of each parity tile, Aubrey has a 3/5 probability of grabbing tiles of opposite parity. Then Eamon will succeed with a probability of 2/3 and Neala will always succeed assuming the other three have. Thus the probability that all four succeed is the product $\frac{4}{7} \cdot \frac{3}{5} \cdot \frac{2}{3} = \frac{24}{105} = \boxed{\frac{8}{35}}$ in lowest terms.

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12. Note that P is 17 units from the center of the circle, the circle has radius 8, and the tangent and radius at a point on a circle are perpendicular. So the distance from P to Q is 15 from the Pythagorean theorem. Knowing these distances, we can set $(x-5)^2 + (y-4)^2 = 8^2$ and $(x-22)^2 + (y-4)^2 = 15^2$. Subtracting the first of these from the second leaves $459 - 34x = 161$. Solving, $x = \boxed{149/17}$.

If you don't like solving equations, another way to find x would be to drop a perpendicular from Q to the horizontal line $y = 4$ between P and the center of the circle. If C is the center and S is the foot of the perpendicular, the triangle CQS is similar to the 8-15-17 triangle CPQ. So $CQ = \frac{8}{17}CR = 64/17$. Since the x-coordinate of C is 5, we add to obtain $5 + 64/17 = 149/17$.

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13. Note that after the neighbor lends the inheritors an emu, there are $n+1$ emus in the flock, and the separation condition now says that $\frac{n+1}{2} + \frac{n+1}{3} + \frac{n+1}{h} + 1 = n+1$. Rewriting, $1 = (n+1)(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{h})$. Simplifying the fractions, $1 = (n+1)(\frac{1}{6} - \frac{1}{h}) = (n+1)\frac{h-6}{6h}$. So we learn that $\frac{6h}{h-6} = n+1$ is an integer.

Rewrite the left-hand side as $\frac{6(h-6+6)}{h-6} = 6 + \frac{36}{h-6}$. This can only be an integer if $h - 6$ is a factor of 36, and the larger h is the smaller this will become. So since we wish n to be as large as possible we should choose h to make $h - 6$ the smallest possible divisor of 36, namely 1. So set $h = 7$ and solve to find $n = \boxed{41}$.

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14. The two-digit squares are 16, 25, 36, 49, 64, and 81. We wish to chain these together so that the units digit of one is the tens digit of the next as long as possible. Inspection shows this to be $\boxed{81649}$.

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15. A lattice point requires both x and y to be integers. That makes the exponent an integer. Of course, as long as $187 - x^2 \geq 0$, an integer to this power will be an integer, so the 27 points where x runs from -13 to 13 all lead to lattice points. If the exponent is a negative integer, then y will be an integer precisely when the base $x - 187$ is ± 1. This requires $x = 188$ or $x = 186$. Altogether this gives $\boxed{29}$ lattice points.

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16. It might be most convenient to convert all logarithms to the same base. We'll use a common base of 2. Using the base change formula, $\log_8 a^2 = \frac{\log_2 a^2}{\log_2 8} = \frac{\log_2 a^2}{3} = \log_2 a^{2/3}$, using the laws of logs and exponents in the last step. Changing the other bases leads to the revised equations $\log_2 a^{2/3} + \log_2 b^{3/2} = \log_2 64$ and $\log_2 a^{3/2} + \log_2 b^{2/3} = \log_2 128$.

Now add both equations, using the laws of logarithms and exponents to yield $\log_2 (ab)^{13/6} = \log_2 2^{13}$. Thus $(ab)^{1/6} = 2$ so $ab = 2^6 = \boxed{64}$.

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17. A number is divisible by 165 precisely when it is divisible by the prime factors of 165, which are 3, 5, and 11. Now a number is divisible by 5 when it ends in 0 or 5 (0 in this case, since our number must consist only of 2's and 0's). It is divisible by 3 when the sum of its digits is divisible by 3. Since the digits are 2's and 0's, the number of 2's must be a multiple of 3. Finally, to be divisible by 11, the alternating digits sums (even-place digits and odd-place digits) must be equal or differ by a multiple of 11. The smallest number we can find that satisfies these requirements has six 2's in it. So we can make the number $\boxed{2{,}222{,}220}$ that satisfies all these requirements, and no smaller number does.

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18. First, the possible values of 5-Across are 2020, 4040, 6060, and 8080. Thus it follows that the middle digit of 2-Down is 0 and likewise for the middle digit of 4-Down. Next, consider 1-Down. The three-digit Fibonacci numbers are 144, 233, 377, 610, and 987. Because the leftmost digit of 5-Across is a positive even digit, the only candidates for 1-Down are 144 and 987. Note that 1-Across is a number of the form $n^3 - n$, for some positive integer n. If 1-Down were 987, then the only possible value for 1-Across would be $21^3 - 21$, as $20^3 - 20 < 8000$ and $22^3 - 22 > 10000$. However, $21^3 - 21$ is a multiple of 10, and leading zeros are not allowed. Thus 1-Down is 144. Furthermore, the only candidates for 2-Down are $5! - 11 = 109$ and $6! - 11 = 709$. Thus the units digit of 2-Down is 9. If 2-Down were 109, then this would imply that there is a four-digit number $\underline{1}\,\underline{1}\,\underline{A}\,\underline{B}$ of the form $n^3 - n$ for some integer n. However, the least such four-digit number is $11^3 - 11 = 1320$; hence 2-Down must be 709. Because $12^3 = 1728$, it follows that the desired value of n for 1-Across is 12 and that 1-Across is $12^3 - 12 = 1716$. It immediately follows that the units digit of 3-Down is 1 and that 6-Across is $17^3 = 4913$. Finally, we can check that 603 indeed satisfies the condition required for 4-Down. Thus the completed puzzle is

1	7	1	6
4	0	4	0
4	9	1	3

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19. There are, of course, 6^5 possible combinations of rolls. We need to determine how many of these have the numbers come out in non-decreasing order. To do this, select any five numbers (with repetitions allowed) from one up to six. Since these must occur in non-decreasing order, we only need to know how many ways there are to make the selection. To do this, place five bars in a row. Then sprinkle amongst these bars five stars. Each star represents a person. The number of stars to the left of the first bar is the number who roll ones; the number of stars between the first and second bars is the number of people who roll twos, and so on. So the string $||**||*|**$ (for example) means no one rolled a one or two, two people rolled threes, no one rolled a four, there was one five, and then the last two people rolled sixes. Now it is easy to count the number of possible selections of numbers, for there are ten symbols, five bars and five stars. So we select the positions for the stars and the bars fall into place. Thus, there are $\binom{10}{5} = 252$ possible selections. To find the probability, we divide by 6^5 to obtain $\boxed{\dfrac{7}{216}}$.

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20. A very "bashy" way to get an answer is simply to apply the tangent to both sides. On the left, $\tan(2\tan^{-1}(x) + \tan^{-1}(2x))$, while on the right $\tan(\frac{\pi}{2}$ is undefined. So the left side must also be undefined. Using that addition formula for tangents, the left side can be expanded to

$$\frac{\tan(2\tan^{-1}(x)) + \tan(\tan^{-1}(2x))}{1 - \tan(2\tan^{-1}(x))\tan(\tan^{-1}(2x))}.$$

To be undefined, the denominator should be zero. Of course, $\tan(\tan^{-1}(2x)) = 2x$, so to make the denominator zero we must have $\tan(2\tan^{-1}(x)) = \frac{1}{2x}$. Using the double-angle formula for tangent and that fact that $\tan(\tan^{-1}(x)) = x$ then yields

$$\frac{2x}{1-x^2} = \frac{1}{2x}.$$

Cross-multiplying, $4x^2 = 1 - x^2$ so $5x^2 = 1$ or $x^2 = \boxed{1/5}$.

A more elegant approach is to use complex numbers. Let $w = 1 + xi$ and $z = 1 + 2xi$. Then the argument (the θ in $r\operatorname{cis}(\theta)$) of w is $\tan^{-1}(x)$, while that of z is $\tan^{-1}(2x)$. Then from the properties of complex multiplication, the argument of w^2z is $2\tan^{-1}(x)+\tan^{-1}(2x)$ which is given to be $\pi/2$. Thus w^2z is a purely imaginary number, so its real part must be zero. Now $w^2z = (1 + xi)^2(1 + 2xi) = (1 + 2xi - x^2)(1 + 2xi) = 1 + 4xi - 5x^2 - 2x^3i$. The real part is $1 - 5x^2$, and setting this to zero instantly delivers the result $x^2 = \boxed{1/5}$.

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21. This is a very standard geometric probability question. Let s and m represent the two times (in minutes after 4 pm) that Sheila and Mary, respectively, arrive. If the point (s, m) is near the diagonal of the square $[0, 60] \times [0, 60]$ they meet, otherwise they do not. The parts of this square where they do not meet—in the upper left where m is much larger than s and the lower right where s is much larger than m—together form a square whose sides are $60 - m$ minutes long.

Since the probability that they do not meet is also $1/2$, we must have $(60-m)^2/3600 = 1/2$ or $(60-m)^2 = 1800$. So $60-m = 30\sqrt{2}$ or $m = \boxed{60-30\sqrt{2}}$.

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22. Let x represent the unknown distance LM. Apply the law of cosines to $\triangle ALR$ to find that $AL^2 = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cdot \cos(120°)$. Since $\cos(120°) = -1/2$, this simplifies to $AL^2 = 25 + 64 + 40 = 129$. If the law of cosines is applied to $\triangle ALM$ the result is $AM^2 = 64+x^2+8x$. Finally, applying the law of cosines to $\triangle RLM$ yields $RM^2 = 25+x^2+5x$.

Now we are also given that $\triangle ARM$ has a right angle at R, so the Pythagorean theorem yields $AR^2 + RM^2 = AM^2$. Inserting the values found in the previous paragraph gives $129 + 25 + x^2 + 5x = 64 + x^2 + 8x$. Collecting like terms yields $90 = 3x$ or $x = \boxed{30}$.

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23. Since $2020 = 2^2 \cdot 5 \cdot 101$, the complementary factor in the smallest cube will be $2 \cdot 5^2 \cdot 101^2 = 50 \cdot 10201 = \boxed{510050}$.

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24. Since $A^2 + R^2 = 20$, (A, R) must be $(2, 4)$ or $(4, 2)$, either yielding the sum $A + R = 6$. On the other hand, $M^2 - L^2 = (M + L)(M - L) = 20$. To make $M + L$ as large as possible, we make $M - L$ as small as possible. Since M and L are both integers, $M + L$ and $M - L$ have the same parity, so we cannot make the factors be 20 and 1, though we can manage 10 and 2 (try $M = 6$ and $L = 4$). So $M + L = 10$. Then $A + R + M + L = \boxed{16}$.